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Effect of a Wavy Bottom on Kelvin-Helmholtz Instability

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The linearized Kelvin-Helmholtz free surface instability is investigated for the case where the upper fluid is bounded and streaming with a velocity *U*, and the lower fluid is at rest and bounded by a sinusoidal bottom. A vortex sheet model is formulated, and a perturbation solution in the bottom amplitude is developed in which the time variable is strained. For any given set of physical parameters, it is found that flat-bottom stability implies wavy-bottom stability, except for a discrete set of exceptional wave numbers, for which no conclusion regarding stability can be drawn from the present analysis.

Nomenclature

A = the difference ξ-x
$\mathbf{A}_r$ = coefficient matrix, see Eq. (26)
B = $\eta(x, t) - \eta(\xi, t)$
$\mathbf{C}_n{}^r$ = column vector, see Eq. (26)
d = mean undisturbed depth
g = vorticity perturbation, see Eq. (8)
g = acceleration of gravity
$\mathbf{g}$ = column vector $(g, h, \bar{g})$ transpose
$\mathbf{g}_j$ = jth coefficient in λ-expansion of $\mathbf{g}$, see Eq. (16)
$\mathbf{G}_n{}^r$ = column vector of g, h, $\bar{g}$ amplitudes, see Eq. (21)
h = free surface perturbation, see Eq. (8)
i = $(-1)^{1/2}$ and dummy index
I_j = velocity perturbations (j = 1, 2, 3), see Eqs. (9-12)
$I_i{}^j$ = jth coefficient in λ-expansion of I_i, see Eq. (18)
${}^rI_i{}^n$ = the part of $I_i{}^n$ involving $\mathbf{g}_r$
k = wave number
p = pressure

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$sgnk$ = + 1 if $k>0$ and -1 if $k<0$
t = time
T = surface tension
u, v = x, y velocity components (total), respectively
U = freestream velocity of upper fluid
U_k = critical velocity for flat bottom ($\lambda = 0$) case, see Eq. (25)
x, y = Cartesian coordinates
β_j = straining coefficients in Eq. (17)
γ = free surface circulation per unit x-length
δ = density ratio ρ^+/ρ^-
ϵ = small parameter, see Eq. (8)
η = free surface deflection
κ = $(\rho^- - \rho^+)/(\rho^- + \rho^+)$
λ = amplitude of wavy bottom
ν = nondimensional parameter, $T/(\rho^- g\, d^2)$
ξ = dummy x-variable
ρ = fluid density
σ = disturbance frequency
τ = strained time variable, see Eq. (17)
ω = frequency of wavy bottom
$(\ )^\pm$ = + and − refers to upper and lower fluids
$(\bar{\ })$ = bar denotes reference to wavy bottom
$(\ )^*$ = * denotes nondimensionization

Introduction

THE classical Kelvin-Helmholtz instability[1,2] occurs at the interface between two plane uniformly streaming flows of incompressible inviscid fluids in the presence of a downward gravity field g and surface tension T. It is found that with respect to a freesurface disturbance of the form exp$i$ (kx − σt) (disturbances parallel to the stream being found to be the most destructive), the flow will be stable if

$$(U^+ - U^-)^2 \le \frac{g}{k}\,\frac{(\rho^-)^2 - (\rho^+)^2}{\rho^+\rho^-} + Tk\,\frac{\rho^+ + \rho^-}{\rho^+\rho^-} \tag{1}$$

where U^+, U^- are the upper and lower freestream velocities and ρ^+, ρ^- are the two (uniform) densities.

Motivated partly by an interest in air-sea interaction phenomena and two-phase flow in ducts, the original analysis was subsequently extended to account for upper and lower plane boundaries[3,4] as well as nonuniform velocity and density distributions.[5-7]

In the present paper we consider instead the effect of a more complicated wall geometry. Specifically, we will consider the fluid to be unbounded above, and bounded below by a sinusoidal bottom of amplitude λ and frequency ω (Fig. 1), where λ need not be small compared to the mean depth d. So that the unperturbed free surface will be flat, we choose $U^- = 0$, and simply call $U^+ \equiv U$.

Although it is not essential, it will be convenient to follow the vortex sheet approach of Zaroodny and Greenberg,[8] whereby the unknowns are the free surface deflection $\eta(x,t)$ and the circulation per unit x-length $\gamma(x,t)$ and $\bar{\gamma}(x,t)$ on the free surface $\mathcal{S}$ and bottom $\bar{\mathcal{S}}$, respectively; this automatically ensures the Laplacian nature of the upper and lower flows, as well as the requirement that the normal velocity be continuous across $\mathcal{S}$. It remains, then, to satisfy the kinematic free surface condition

$$\eta_t + u\eta_x = v \tag{2}$$

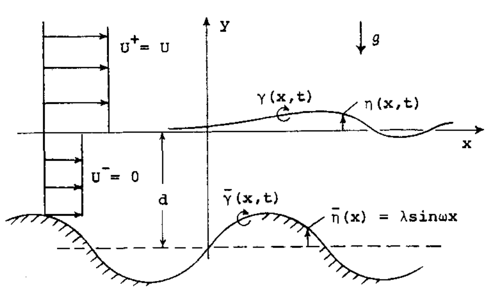

Fig. 1 Geometry and vortex sheets.

on $\mathcal{S}$, the dynamic condition

$$p^+ - p^- = T\eta_{xx}(1 + \eta_x^2)^{-3/2}$$

or [as developed in Ref. 8]

$$\gamma_t + (u\gamma)_x = \frac{\kappa}{2}[C^2\gamma(\gamma_x - CS\gamma\eta_{xx}) + 4(u_t + uu_x) + 4\eta_x(v_t + uv_x) + 4g\eta_x] - \frac{2T}{\rho^+ + \rho^-}\left\{\frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}\right\}_x \tag{3}$$

on $\mathcal{S}$, and the tangent flow condition

$$\bar{\gamma} = 2\bar{u}(1 + \bar{\eta}_x^2) \tag{4}$$

on $\bar{\mathcal{S}}$, where u,v are the x,y velocity components, p is the pressure, $\kappa = (\rho^- - \rho^+)/(\rho^- + \rho^+)$, $\bar{u}$ is the x-velocity on $\bar{\mathcal{S}}$, and C,S are short for $\cos(\tan^{-1}\eta_x)$, $\sin(\tan^{-1}\eta_x)$, respectively. Equations (2-4) are three equations in the three unknowns $\gamma, \eta, \bar{\gamma}$ where u,v on $\mathcal{S}$, and $\bar{u}$ on $\bar{\mathcal{S}}$ are given by

$$u(x,t) = \frac{U}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{\gamma B}{A^2 + B^2} + \frac{\bar{\gamma}[d + \eta(x,t) - \bar{\eta}(x)]}{A^2 + [d + \eta(x,t) - \bar{\eta}(x)]^2}\right\}d\xi \tag{5}$$

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{\gamma A}{A^2 + B^2} + \frac{\bar{\gamma}A}{A^2 + [d + \eta(x,t) - \bar{\eta}(x)]^2}\right\}d\xi \tag{6}$$

$$\bar{u}(x,t) = \frac{U}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{\bar{\gamma}\bar{B}}{A^2 + \bar{B}^2} - \frac{\gamma[d + \eta(x,t) - \bar{\eta}(x)]}{A^2 + [d + \eta(x,t) - \bar{\eta}(x)]^2}\right\}d\xi \tag{7}$$

with $A = \xi - x$, $B = \eta(x,t) - \eta(\xi,t)$ and $\bar{B} = \bar{\eta}(x) - \bar{\eta}(\xi)$; the dummy variable ξ is understood where omitted, e.g., γ denotes $\gamma(\xi,t)$ rather than $\gamma(x,t)$.

Nondimensionalization

We nondimensionalize all velocities, γ and $\bar{\gamma}$ with respect to $(gd)^{1/2}$, all lengths with respect to the mean depth d, and time with respect to $(d/g)^{1/2}$. Thus,

$$\gamma^* = \gamma/(gd)^{1/2},\ g^* = g/(gd)^{1/2},\ u^* = u/(gd)^{1/2},\ v^* = v/(gd)^{1/2},$$

$$x^* = x/d,\ \xi^* = \xi/d,\ \eta^* = \eta/d,\ A^* = A/d,\ B^* = B/d,$$

$$\lambda^* = \lambda/d,\ \omega^* = \omega d,\ t^* = (g/d)^{1/2}t$$

and so on. *From here on the asterisks will be dropped, for simplicity, and all quantities will be understood to be nondimensionalized.*

The parameters that arise are

$$\delta \equiv \rho^+/\rho^- \text{ and } \nu \equiv T/(\rho^- g d^2)$$

Stability Analysis

To investigate the infinitesimal stability we perturb the base values $\gamma = U$, $\eta = \bar{\gamma} = 0$ according to

$$\left.\begin{aligned}\gamma &= U + \epsilon g(x,t)\\ \eta &= \epsilon h(x,t)\\ \bar{\gamma} &= \epsilon\bar{g}(x,t)\end{aligned}\right\} \tag{8}$$

To within $0(\epsilon)$,

$$u = U/2 + \epsilon I_1,\ v = \epsilon I_2,\ \bar{u} = \epsilon I_3 \tag{9}$$

where

$$I_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{U[h(x,t) - h]}{A^2} + \frac{\bar{g}(1 - \lambda \sin\omega\xi)}{A^2 + (1 - \lambda \sin\omega\xi)^2}\right\}d\xi \tag{10}$$

$$I_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{g}{A} + \frac{\bar{g}A}{A^2 + (1 - \lambda \sin\omega\xi)^2}\right\}d\xi \tag{11}$$

$$I_3 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{2Uh(1 - \lambda \sin\omega x)^2}{[A^2 + (1 - \lambda \sin\omega x)^2]^2} - \frac{Uh + g(1 - \lambda \sin\omega x)}{A^2 + (1 - \lambda \sin\omega x)^2} + \frac{\lambda\bar{g}(\sin\omega x - \sin\omega\xi)}{A^2 + \lambda^2(\sin\omega x - \sin\omega\xi)^2}\right\}d\xi \tag{12}$$

Inserting Eqs. (8) and (9) into Eqs. (2–4) and linearizing in ϵ yields the equations

$$g_t + \frac{U}{2}(1 - \kappa)g_x - 2\kappa h_x + \left(\frac{2\nu}{1 + \delta}\right)h_{xxx} = 2\kappa(I_1)_t - U(1 - \kappa)(I_1)_x \tag{13}$$

$$h_t + \frac{U}{2}h_x = I_2 \tag{14}$$

$$\bar{g} = 2I_3(1 + \bar{\eta}_x^{\,2}) \tag{15}$$

on the three perturbation quantities. To solve, we express

$$\mathbf{g} = \mathbf{g}_0(x,\tau) + \lambda\mathbf{g}_1(x,\tau) + \lambda^2\mathbf{g}_2(x,\tau) + \ldots \tag{16}$$

$$\tau = (1 + \beta_2\lambda^2 + \beta_4\lambda^4 + \ldots)t \tag{17}$$

where $\mathbf{g}$ is the column vector $(g,h,\bar{g})$ transpose and τ is a strained time variable, the straining being needed to regularize the perturbation scheme. We have anticipated that $\beta_1 = \beta_3 = \ldots = 0$; this will be explained following Eq. (22).

Of course we need to expand I_1, I_2, I_3 in terms of λ

$$I_i = I_i^{\,0} + \lambda I_i^{\,1} + \lambda^2 I_i^{\,2} + \ldots \tag{18}$$

A typical term I_i^n involves components of $\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_n$. Letting ${}^rI_i^n$ denote the part of I_i^n involving $\mathbf{g}_r$, we have

$$I_i^{\,n} = \sum_{r=0}^{n} {}^rI_i^{\,n} \tag{19}$$

Inserting Eqs. (16–18) into Eqs. (13–15) and equating powers of λ leads to the final equations on $\mathbf{g}_n$

$$g_{n\tau} + \frac{U}{2}(1 - \kappa)g_{nx} - 2\kappa h_{nx} + \left(\frac{2\nu}{1 + \delta}\right)h_{nxxx} - \left\{2\kappa\frac{\partial}{\partial\tau} - U(1 - \kappa)\frac{\partial}{\partial x}\right\}({}^nI_1^{\,n}) = \left\{2\kappa\frac{\partial}{\partial\tau} - U(1 - \kappa)\frac{\partial}{\partial x}\right\}\sum_{r=0}^{n-1} {}^rI_1^{\,n} + [\beta_2(I_1^{\,n-2} - g_{n-2}) + \ldots + \beta_i(I_1^{\,n-i} - g_{n-i})]_\tau \tag{20a}$$

$$h_{n\tau} + \frac{U}{2}h_{nx} - {}^nI_2^{\,n} = \sum_{r=0}^{n-1} {}^rI_2^{\,n} - [\beta_2 h_{n-2} + \ldots + \beta_i h_{n-i}]_\tau \tag{20b}$$

$$\bar{g}_n - 2({}^nI_3^{\,n}) = 2\sum_{r=0}^{n-1} {}^rI_3^{\,n} + 2\omega^2 I_3^{\,n-2}\cos^2\omega x \tag{20c}$$

where $i = n$ if n is even, and $n - 1$ if n is odd.

As is customary, we impose discrete disturbances of the form $\exp i(kx - \sigma\tau)$ on the flow

$$\mathbf{g}_n = \sum_{j=-n}^{n} \mathbf{G}_n^{\,j} \exp i[(k + j\omega)x - \sigma\tau] \tag{21}$$

where the additional $j\omega$ harmonics are necessitated by the forcing terms on the right hand side of Eq. (20). These disturbances will be stable if and only if

$$Im[\sigma(1 + \beta_2\lambda^2 + \ldots)] \le 0 \tag{22}$$

Observe that changing the sign of λ is merely equivalent to translating the x,y coordinate system to the right or left by π/ω, and surely this cannot affect the stability. Thus, odd powers of λ should not be present in Eq. (22), and this is why we omitted them back in Eq. (17).

For $n = 0$ the system Eq. (20) is homogeneous, and insertion of Eq. (21) leads to the matrix equation

$$\mathbf{A}_0\mathbf{G}_0^{\,0} = \mathbf{O} \tag{23}$$

where $\mathbf{A}_0$ is the matrix

$$\begin{pmatrix} -\frac{2\sigma}{k} + (1 - \kappa)U & \begin{matrix} U^2(1 - \kappa)|k| + 2\kappa U\sigma\,\mathrm{sgn}k \\ -4\kappa - 4\nu k^2/(1 + \delta) \end{matrix} & [\frac{2\kappa\sigma}{k} + (1 - \kappa)U]e^{-|k|} \\ \mathrm{sgn}k & 2\sigma - Uk & e^{-|k|}\mathrm{sgn}k \\ e^{-|k|} & -U|k|e^{-|k|} & 1 \end{pmatrix}$$

The requirement that $\det\mathbf{A}_0 = 0$ provides a quadratic equation in σ. Solving, we find that the allowable σ's, corresponding to the wave number k, are given by

$$\frac{\sigma}{k} = \frac{\delta U \pm \{(U_k^{\,2} - U^2)\delta \coth|k|\}^{1/2}}{\coth|k| + \delta} \tag{24}$$

where

$$U_k^{\,2} \equiv \frac{(\coth|k| + \delta)(1 - \delta + \nu k^2)}{\delta|k|\coth|k|} \tag{25}$$

Results

From Eq. (24), we see that for the flat bottom case, where $\lambda = 0$, the stability condition Eq. (22) reduces simply to $U \le U_k$, in agreement with Refs. 3 and 4. We might just note that the effect of the bottom is destabilizing, with the minimum U_k occurring (for a given k) as $d \to 0$, as could be seen if we wrote the dimensional version of Eq. (25). For $\lambda \ne 0$, of course, we need to know the β_j's. We therefore return to Eq. (20) and consider $n = 1,2,\ldots$.

For each n, equating harmonics breaks the system into a set of matrix equations

$$\mathbf{A}_r\mathbf{G}_n^{\,r} = \mathbf{C}_n^{\,r}, \quad r = 0, \pm 1, \ldots, \pm n \tag{26}$$

For the scheme to remain regular it is sufficient that the matrices $\mathbf{A}_r$ be nonsingular for $r \ne 0$ and, since we already know that zero is an eigenvalue of $\mathbf{A}_0$, it is sufficient that $\mathbf{C}_n^0$ be orthogonal to the eigenvector $\mathbf{G}_0^0$ for each $n = 1,2,\ldots$. If we denote any two rows (say the second and third) of $\mathbf{A}_0$ as $\mathbf{P}$ and $\mathbf{Q}$, then $\mathbf{G}_0^0$ must be proportional to $\mathbf{P} \times \mathbf{Q}$, so that we need

$$\mathbf{C}_n^{\,0}\cdot\mathbf{G}_0^{\,0} = \mathbf{C}_n^{\,0}\cdot\mathbf{P} \times \mathbf{Q} = 0 \quad \text{for } n = 1, 2, \ldots \tag{27}$$

This will be found to be satisfied identically for $n = 1,3,5,\ldots$; for $n = 2,4,\ldots$ it provides the recipe for evaluating $\beta_2, \beta_4, \ldots$, in turn.

Whereas the details of the calculation are tedious (Appendix), we find that $\mathbf{C}_n^0$ is a linear combination of $\mathbf{G}_{n-1}^p, \mathbf{G}_{n-3}^p, \ldots$ terms with imaginary coefficients, and $\mathbf{G}_{n-2}^p, \mathbf{G}_{n-4}^p, \ldots$ terms with real coefficients, and that $\mathbf{G}_0^p$, $\mathbf{G}_1^p, \ldots$ are alternately purely real and imaginary. It follows that the left hand side of Eq. (27) either contains no i's at all, or else contains an i factor in each term—which therefore cancel. Thus Eq. (27) yields β_n as a real-valued function of σ, k, the previous β_j's (i.e., $j < n$), and the various physical parameters. We conclude that if σ is real

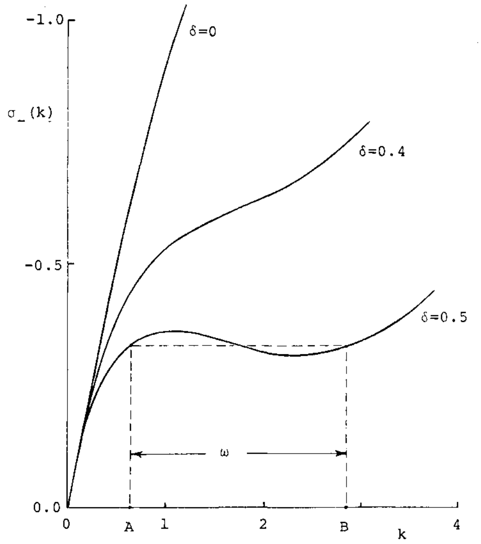

Fig. 2 $\sigma_{-}(k)$ **for** $U = 1/(2)^{1/2}$ **and** $\nu = 0.05$.

then all the β_n's are too, so that *flat-bottom stability implies wavy-bottom stability.* More precisely, if for a given wave number k the velocity U is such that the free surface is stable in the presence of a flat bottom at depth 1, then it will also be stable for a wavy bottom $\bar{\eta} = \lambda \sin\omega x$ at an average depth 1, for all ω and λ (of course $|\lambda| < 1$).

Actually, before drawing this conclusion we were supposed to also check to see that the $\mathbf{A}_r$'s are all nonsingular, for $r = \pm 1, \pm 2, \ldots$. Expressing

$$det\mathbf{A}_0 = a(k)\sigma^2(k) + b(k)\sigma(k) + c(k) = 0 \qquad (28)$$

it is found that

$$det\mathbf{A}_r = a(k + r\omega)\sigma^2(k) + b(k + r\omega)\sigma(k) + c(k + r\omega) \qquad (29)$$

Replacing k in Eq. (28) by $k + r\omega$ and subtracting from Eq. (29) yields the more convenient form

$$det\mathbf{A}_r = [\sigma(k) - \sigma(k + r\omega)]\{a(k + r\omega)[\sigma(k) + \sigma(k + r\omega)] + b(k + r\omega)\} \qquad (30)$$

where

$$\left.\begin{aligned} a(k) &= -4(1 + \kappa e^{-2|k|})/k \\ b(k) &= 4U(1 - \kappa)(1 - e^{-2|k|}) \end{aligned}\right\} \qquad (31)$$

Considering $\delta \to 0$, for simplicity, we see that $b \to 0$, $a \neq 0$, and

$$\sigma(k) = \pm\, sgnk(|(k + \nu k^3)\tanh k|)^{1/2} \equiv \sigma_{\pm}(k) \qquad (32)$$

from Eq. (24). σ_+ corresponds to right-running waves, and σ_- to left-running ones; both cases are possible.

Now, σ_+ is odd and monotone increasing, and σ_- is odd and monotone decreasing. Consequently, whereas the $\sigma(k) - \sigma(k + r\omega)$ factor must be nonzero, the $\sigma(k) + \sigma(k + r\omega)$ factor will indeed be zero for $k = r\omega/2$ for $r = \pm 1, \pm 2, \ldots$.

Thus, for the case where $\delta \to 0$ we must qualify our previous claim that flat-bottom stability implies wavy-bottom stability. That is, it is valid for all k which are not integer multiples of $\omega/2$; for these exceptional wave numbers our perturbation scheme is singular, and no conclusion regarding stability can be drawn from the present analysis.

The presence of a set of exceptional wave numbers is found to persist for $\delta > 0$ as well, not only due to zeros of the curly bracket in Eq. (30) but also because $\sigma_{-}(k)$ eventually develops a kink as δ increases (Fig. 2), so that the first factor in Eq. (30) may also be zero at certain k's (A and B, i.e. in Fig. 2) if ω is not too large.

Appendix

In the paragraph following Eq. (27) we summarized the results of calculations which we felt were too tedious to include in the main body of the paper, and which led to our claim that "if σ is real then all the β_n's are too." Here we outline these calculations.

Of the three Eqs. (20), consider for example Eq. (20b):

$$h_{n\tau} + \frac{U}{2}h_{nx} - {}^{n}I_2{}^{n} = \sum_{r=0}^{n-1} {}^{r}I_2{}^{n} - \sum_{\substack{0 \le r < n \\ \text{for } n-r \text{ even}}} \beta_{n-r}h_{r\tau} \qquad (A1)$$

We will need to know only the *form* of these various terms, especially whether they are real or imaginary, and this simplifies our task.

Let us denote any *real* linear combination of terms of the form $G_l^s \exp i[(k + p\omega)x - \sigma\tau] = G_l^s \exp \alpha_p(x,\tau)$ in which $|p| \le n$, $0 \le l \le n$, and $|s| \le l$ by $L(l,n;x)$; the τ dependence is not especially relevant here. G_l^2 stands for any of the three components of the $\mathbf{G}_l^2$ vector. For example, recalling from Eq. (21) that

$$h_n(x,\tau) = \sum_{j=-n}^{n} G_n{}^j \exp\alpha_j(x,\tau)$$

we see that $h_n(x,\tau) = L(n,n;x)$, and that

$$h_{n\tau}(x,\tau),\, h_{nx}(x,\tau) = iL(n,n;x) \qquad (A2)$$

for the first two terms in Eq. (A1). (Again, L denotes the *form* of the quantity; two quantities having the same L representation need not be identical.) Similarly, the last term in Eq. (A1) is

$$\sum_{\substack{0 \le r < n \\ \text{for } n-r \text{ even}}} \beta_{n-r}h_{r\tau} = \sum_{\substack{0 \le r < n \\ \text{for } n-r \text{ even}}} iL(r,r;x) \qquad (A3)$$

if we tentatively assume that all the β's are real.

Finally, we examine the form of the two I terms in Eq. (A1). Recall Eq. (11). The coefficient $\phi(q)$, say, of λ^q in the expansion of $[A^2 + (1 - \lambda \sin\omega\xi)^2]^{-1}$ is

$$\phi(q) = \sin^q\omega\xi \text{ times a function of } A^2. \qquad (A4)$$

Thus the integrand of I_2, see Eq. (11), is

$$\frac{1}{A}\sum_{r=0}^{\infty} g_r(\xi,\tau)\lambda^r + A\left[\sum_{r=0}^{\infty} \bar{g}_r(\xi,\tau)\right]\left[\sum_{q=0}^{\infty}\phi(q)\lambda^q\right]$$

in which the coefficient of λ^n is

$$\frac{1}{A}g_n(\xi,\tau) + A\sum_{r=0}^{n}\bar{g}_r(\xi,\tau)\phi(n-r)$$

so that

$$2\pi I_2{}^n = \int_{-\infty}^{\infty}\left[\frac{1}{A}g_n(\xi,\tau) + A\sum_{r=0}^{n}\bar{g}_r(\xi,\tau)\phi(n-r)\right]d\xi \qquad (A5)$$

Letting θ, E denote any real-valued odd and even functions of $\xi - x$, observe from Eq. (A4) that $\phi(q) = (\sin^q\omega\xi)E$. With the $1/A$ and A both equal to θ, and $\theta E = \theta$, we have from Eqs. (A5) and (20),

$${}^{r}I_2{}^{n} = \int_{-\infty}^{\infty}[g_r(\xi,\tau)\delta_{nr} + \bar{g}_r(\xi,\tau)\sin^{n-r}\omega\xi]\theta d\xi$$

where δ_{nr} is the Kronecker delta. Now, in the product ($n \ge r \ge 0$)

$$\bar{g}_r(\xi,\tau)\sin^{n-r}\omega\xi = \left(\frac{e^{i\omega\xi} - e^{-i\omega\xi}}{2i}\right)^{n-r}\sum_{j=-r}^{r} G_r{}^j e^{i[(k+j\omega)\xi - \sigma\tau]}$$

The exponents range from $-i(n - r)\omega\xi + i[(k - r\omega)\xi - \sigma\tau]$ to $i(n - r)\omega\xi + i[(k + r\omega)\xi - \sigma\tau]$, i.e., from $\alpha_{-n}(\xi,\tau)$ to $\alpha_n(\xi,\tau)$. Thus

$$\bar{g}_r(\xi,\tau)\sin^{n-r}\omega\xi = i^{n-r}L(r,n;\xi)$$

so

$$ {}^rI_2{}^n = \int_{-\infty}^{\infty} [L(r,r,\xi)\delta_{nr} + i^{n-r}L(r,n;\xi)]\theta d\xi $$

$$ = i^{n-r}\int_{-\infty}^{\infty} L(r,n;\xi)\theta d\xi \qquad \text{(A6)} $$

But $\int_{-\infty}^{\infty} e^{i[(k+p\omega)\xi-\sigma\tau]}\theta(\xi - x)d\xi = \int_{-\infty}^{\infty} e^{i[(k+p\omega)(\zeta+x)-\sigma\tau]}\theta(\zeta)d\zeta$

$$ = e^{i[(k+p\omega)x-\sigma\tau]}\int_{-\infty}^{\infty} e^{i(k+p\omega)\zeta}\theta(\zeta)d\zeta $$

$$ = ie^{i[(k+p\omega)x-\sigma\tau]}\int_{-\infty}^{\infty} \sin(k + p\omega)\zeta\theta(\zeta)d\zeta $$

so

$$ \int_{-\infty}^{\infty} L(r,n;\xi)\theta d\xi = iL(r,n;x) $$

and Eq. (A6) becomes

$$ {}^rI_2{}^n = i^{n-r+1}L(r,n;x) \qquad \text{(A7)} $$

With Eqs. (A2, A3, and A7), Eq. (A1) becomes

$$ iL(n,n;x) + iL(n,n;x) - i^{n-n+1}L(n,n;x) $$

$$ = \sum_{r=0}^{n-1} i^{n-r+1}L(r,n;x) - \sum_{\substack{0\le r<n \\ \text{for } n-r \text{ even}}} iL(r,r;x) \qquad \text{(A8)} $$

Looking at the last term on the right, note that with $r < n$ the range of α_p's in $L(r,r;x)$ is included in the range of α_p's in $L(r,n;x)$, so that $L(r,r;x)$ in Eq. (A8) can be replaced by $L(r,n;x)$. Furthermore, i can be changed to i^{n-r+1} since $n - r$ is even. Thus the two terms on the right side of Eq. (A8) can be combined, and Eq. (A8) becomes

$$ L(n,n;x) = \sum_{r=0}^{n-1} i^{n-r}L(r,n;x) $$

or

$$ L(n,n;x) = \sum_{\substack{0\le r<n \\ \text{for } n-r \text{ even}}} L(r,n;x) + i\sum_{\substack{0\le r<n \\ \text{for } n-r \text{ odd}}} L(r,n;x) \qquad \text{(A9)} $$

It turns out that the other Eqs., (20a) and (20c), have the same form as this. Observe that the left hand side of Eq. (A9), and hence of Eqs. (20) and (26), have real coefficients for the $\mathbf{G}_n{}^p$ components, and the right hand sides (i.e., the inhomogeneous terms) have imaginary coefficients for $\mathbf{G}_{n-1}{}^p, \mathbf{G}_{n-3}{}^p, \ldots$, and real coefficients for $\mathbf{G}_{n-2}{}^p, \mathbf{G}_{n-4}{}^p, \ldots$, as claimed in the paragraph following Eq. (27).

Finally, the claim that $G_0{}^p, G_1{}^p, \ldots$ are alternately purely real and imaginary can be proved by induction, starting with the fact that $\mathbf{G}_0{}^0$ is real (since the $\mathbf{A}_0$ matrix is real and Eq. (23) is homogeneous, so $\mathbf{G}_0{}^0$ is an arbitrary constant times a real vector, which we can assume to be real), and using the information contained in the preceding sentence.

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